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STATIONARY WAVES IN A HOMOGENEOUS MEDIUM PERTURBED BY AN INCLUSION IN THE FORM OF A CURVILINEAR ROD*

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A three-dimensional homogeneous medium with an inclusion in the shape of a curvilinear rod is considered. Stationary fields of different kinds in such a medium can be represented in the form of asymptotic expansions in series of a small parameter, the ratio between the characteristic transverse dimension of the inclusion and its length. The paper is devoted to the construction of the principal term of the expansions mentioned on the basis of the integral equation of the problem. The field within the inclusion, in terms of which the field in the medium is expressed by using a known integral operator, turns out to be the main unknown here. It is shown that the principal terms of the asymptotic expansion of the field within the inclusion is represented in the form of a sum of components slowly varying along the rod and boundary-layer type functions localized in the neighbourhood of the rod ends. An equation is obtained for the boundary layer functions and the nature of their damping is investigated as they recede from the ends of the rod. The form of the slowly varying part of the desired asymptotic form depends on the magnitude of the second dimensionless parameter of the problem, the ratio between the physical characteristics of the inclusion and the medium. The field with scalar potential is investigated in detail, the results of applying the method to a field with a vector potential (the theory of elasticity) are presented in the concluding part of the paper.

1. Formulation of the problem. A number of stationary problems in mathematical physics for an inhomogenous medium reduces to determing the scalar or vector potential $u(x)$ and two tensor functions of the field intensity type $\varepsilon(x)$ and flux type $\sigma(x)$ from the following system of linear differential equations ($x(x_1, x_2, x_3)$ is a point of the medium):

$$\operatorname{div}\sigma(x) = -q(x),\quad \sigma(x) = c(x)\cdot\varepsilon(x),\quad \varepsilon(x) = \nabla u(x) \tag{1.1}$$

Here $q(x)$ is the density of the field sources, $c(x)$ is the tensor of the properties of the medium, and convolution of the tensors in one subscript (scalar potentials) or two subscripts (vector potentials) is denoted by a dot. In problems with a scalar potential (stationary heat and electrical conduction, electrostatics, etc.) $c(x)$ is a bivalent tensor. In the case of elasticity theory, $c(x)$ is a quadrivalent tensor of the elastic moduli of the medium, here $u(x)$ is a vector potential while the operator ∇ in (1.1) should be replaced by a symmetrized gradient.

We consider an infinite homogeneous medium with properties tensor c_0 in which there is an isolated inclusion with the properties $c_0 + c_1$. It is assumed that the inclusion is bonded to the medium along the boundary of an ideally and distortion-free internal metric. Let the domain V occupied by the inclusion have the shape of a long curvilinear rod with middle line Γ and circular transverse section of radius $a(z), z \in \Gamma$. Since the transverse dimension of the domain V is substantially less than its length, the function $a(z)$ allows of the representation $a(z) = \varepsilon\, l(z)$, where ε is a small dimensionless parameter, and $l(z)$ is a quantity of the order of the length of the rod. Later we shall assume that Γ is a smooth curve without points of selfintersection, while the function $a(z)$ satisfies the condition $|da/dz| \ll 1$ everywhere on Γ with the exception, perhaps, of the neighbourhoods of the ends of the rod.

We will transfer from system (1.1) to an equivalent integral equation for the field flux, the function $\sigma(x)$ /1, 2/

$$\sigma(x) - \int_V S(x-x')\cdot B_1\cdot\sigma(x')\,dx' = \sigma_0(x) \tag{1.2}$$

$$B_1 = B - B_0,\ B = c^{-1},\ B_0 = c_0^{-1}$$

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Here $\sigma_0(x)$ is the field in the medium in the absence of inhomogeneities but the same sources $q(x)$ and conditions at infinity (external field). The kernel $S(x)$ of the integral operator S in (1.2) is expressed in terms of the second derivatives of Green's function $G(x)$ of a homogeneous medium c_0 and has the form ($\delta(x)$ is the delta function)

$$S(x) = c_0\cdot K(x)\cdot c_0 - c_0\delta(x),\ K(x) = -\nabla\nabla G(x)$$
$$(\nabla\cdot c_0\cdot\nabla G(x) = -\delta(x))$$

Henceforth we will assume that the explicit expression for the function $S(x)$ is known. The integral operator S with kernel $S(x)$ can be considered as a pseudodifferential operator whose symbol $S^*(k)$ (the transformation of the Fourier function $S(x)$) is a homogeneous function of zero degree in $k(k^1, k^2, k^3)$. It is seen from (1.2) that the field $\sigma(x)$ outside the domain V is retored single-valuedly in the values of $\sigma(x)$ within V. Consequently, the field $\sigma(x)$ within the rod can be considered as the fundamental unknown of the problem. We obtain the equation for this field by multiplying both sides of (1.2) by $V(x)$, the characteristic function of the domain V. The solution of such an equation exists and is unique if the determinant of the operator symbol on the left side of (1.2) $\det[I - S^*(k)\cdot B_1]$ does not equal zero for all k (I is the unit tensor) /3/. This condition is satisfied if c is the non-degenerate tensor $\det c \neq 0, \infty$.

The purpose of the paper is to construct the principal term of the expansion of the field $\sigma(x)$ in the domain V in a series in the parameter ε. We will start with an investigation of the passage to the limit as $\varepsilon\to 0$ in the Eq. (1.2).

2. Field with a scalar potential. We will consider the simplest case of a field with a scalar potential when the medium and the inclusion are isotropic. Here $\sigma(x)$ is a vector field $c_0^{\alpha\beta} = c_0\delta^{\alpha\beta}$, $c^{\alpha\beta} = c\delta^{\alpha\beta}$, and the operator symbol S in (1.2) has the form

$$S^{*\alpha\beta}(k) = c_0(k^\alpha k^\beta/k^2 - \delta^{\alpha\beta}) \tag{2.1}$$

where $\delta^{\alpha\beta}$ is the Kronecker delta, and c_0, c are scalar quantities.

We will place the origin of a Cartesian system y_1, y_2, y_3 at an arbitrary point $z\in\Gamma$ ($z\neq z_0$, $z\neq z_l$; z_0, z_l are coordinates of the ends of the rod), by directing the y_3 axis along the tangent to Γ. We change to dimensionless coordinates $\zeta_i = a^{-1}(z)\,y_i$ ($i = 1, 2, 3$) in (1.2) and we allow the parameter ε, and therefore the radius $a(z)$ of the rod also, to tend to zero. The domain V here goes over into V_0, the domain within a cylinder of unit radius with a generator parallel to ζ_3, and (1.2) takes the form

$$\sigma^\circ(z,\zeta) - \int_{V_0} S(\zeta-\zeta')\cdot B_1\cdot\sigma^\circ(z,\zeta')\,d\zeta' = \sigma_0(z),\quad \zeta\in V_0 \tag{2.2}$$

It is here taken into account that $S(x)$ is a homogeneous function of degree -3, while $\sigma^\circ(z,\zeta) = \lim\sigma(y)$ as $\varepsilon\to 0$. Since σ_0 is a vector constant in ζ, then the solution of this equation will also be constant within the domain V_0 and the expression for $\sigma^\circ(z)$ takes the form /4/

$$\sigma^\circ(z) = \Lambda(z)\cdot\sigma_0(z),\ \Lambda^{-1} = I - D\cdot B_1 \tag{2.3}$$
$$D = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{\Omega_1} S^*(k^1, k^2, \varepsilon k^3)\,d\Omega$$

Here Ω_1 is the surface of a unit sphere in k-space (the k^1, k^2, k^3 system of coordinates is conjugate to y_1, y_2, y_3). Substituting $S^*(k)$ from (2.1) here, we obtain

$$\Lambda_\beta{}^\alpha = \frac{2}{2+c_0B_1}\Theta_\beta{}^\alpha + \frac{1}{1+c_0B_1}m^\alpha m_\beta,\quad \Theta_\beta{}^\alpha = \delta_\beta{}^\alpha - m^\alpha m_\beta \tag{2.4}$$

where $m = m(z)$ is the unit vector tangent to Γ at the point z.

It can be expected that, far from the rod ends, the function $\sigma^\circ(z)$ of the form (2.3), (2.4), which is constant in transverse sections of the domain V, will be the principal term of the expansion of the field $\sigma(x)$ in an asymptotic series in the parameter ε. Let us confirm this assumption as follows. We substitute the expression obtained for $\sigma^\circ(z)$ into the left-hand side of (1.2) and we find the residual on the right-hand side. To compensate the residual a certain component $\sigma_1(x)$ should be added to $\sigma^\circ(z)$. If the function $\sigma_1(x)$ tends to zero uniformly in the domain V as $\varepsilon\to 0$, then $\sigma^\circ(z)$ actually is the principal term of the expansion of the field $\sigma(x)$ in a series in ε. If the condition that $\sigma_1(x)$ tend uniformly to zero in V is violated, then the desired asymptotic form is the sum of the function $\sigma^\circ(z)$ and the principal term of the expansion of $\sigma_1(x)$ in a series in ε.

3. Action of the operator $I - SB_1$ on $\sigma^\circ(z)$. We substitute the function $\sigma^\circ(z)$ of the form (2.3), (2.4) into the left-hand side of (1.2) and integrate over the rod transverse section $\omega(z)$. Taking into account that each point $x \in V$ allows of a unique representation in the form $x = z + \rho$, $z \in \Gamma$, $\rho \in \omega(z)$, we will have

$$\sigma^\circ(z) - (SB_1 \cdot \sigma^\circ)(x) = \sigma^\circ(z) - \int_\Gamma \bar{S}(x, z', a) \cdot B_1 \cdot \sigma^\circ(z')\, d\Gamma' \tag{3.1}$$

$$\bar{S}(x, z', a) = \int_{\omega(z')} S(x - z' - \rho')\, d\Omega_{\rho'}, \quad a = a(z') \tag{3.2}$$

The expression for the function $\bar{S}(x, z', a)$ can be represented in the form

$$\bar{S}(x, z', a) = (2\pi)^{-3} \int S^*(k) \exp[-ik \cdot (x - z')]\, dk \int_{\omega(z')} e^{ik \cdot \rho'}\, d\Omega_{\rho'} \tag{3.3}$$

where $S^*(k)$ is the Fourier transform of $S(x)$. We here substitute $S^*(k)$ from (2.1) and integrate first with respect to ρ' and then k^1, k^2. Then in the y_1, y_2, y_3 coordinate system with origin at the point z' and axes directions $e^{(1)}, e^{(2)}, m$, respectively, the expression for $\bar{S}(x, z', a)$ takes the form ($H(t)$ is the Heaviside function)

$$\bar{S}(x, z', a) = \bar{S}(y, z', a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{S}^*(\bar{y}, k^3, z', a) \exp(-ik^3 y_3)\, dk^3 \tag{3.4}$$

$$\begin{gathered} \bar{S}^{*\alpha\beta} = c_0 [\,|\bar{y}k^3|^{-1} S_1 \Theta^{\alpha\beta} + S_2 e^\alpha e^\beta - i \operatorname{sign} k^3 S_1 (m^\alpha e^\beta + e^\alpha m^\beta) - \\ S_0 m^\alpha m^\beta - H(a - |\bar{y}|)\, \Theta^{\alpha\beta}] \quad (\bar{y} = y_1 e^{(1)} + y_2 e^{(2)},\ e = \bar{y}/|\bar{y}|) \\ S_n = \begin{cases} a\,|k^3|\, I_n(|\bar{y}k^3|)\, K_1(a\,|k^3|), & |\bar{y}| < a \\ (-1)^{n+1} a\,|k^3|\, I_1(a|k^3|)\, K_n(|\bar{y}k^3|), & |\bar{y}| > a \end{cases}, \quad n = 0, 1, 2 \end{gathered} \tag{3.5}$$

where I_n, K_n are modified Bessel functions. We examine the asymptotic form of the right side of (3.1) as $a \to 0$. Expanding the functions I_n and K_n in a power series /5/, it can be shown that as $a \to 0$

$$\bar{S}^{*\alpha\beta}(\bar{y}, k^3, z', a) \to \bar{S}_0^{*\alpha\beta}(z') = -\tfrac{1}{2} c_0 [\Theta^{\alpha\beta}(z') + 2m^\alpha(z')\, m^\beta(z')]$$

But then $\lim \bar{S}(y, z', a) = \bar{S}_0^*(z')\, \delta(y_3)$ as $a \to 0$ by virtue of (3.4), and the function $\bar{S}(y, z', a)$ is a δ-sequence /6/. Therefore as $a \to 0$ the integral

$$J(z, a) = \int_\Gamma \bar{S}(z, z', a) \cdot B_1 \cdot \sigma^\circ(z')\, d\Gamma'$$

converges to $\bar{S}_0^*(z) \cdot B_1 \cdot \sigma^\circ(z)$ uniformly on the curve Γ, with the exception of the neighbourhoods of the ends, for any smooth function $\sigma^\circ(z)$. Hence, and from (2.3), (2.4) it follows that as $a \to 0$ the right-hand side of (3.1) converges to $\sigma_0(z)$, the value of the right-hand side of (1.2) on Γ, uniformly in the domain V with the exception of the neighbourhoods of the rod ends.

The convergence of $J(z, a)$ to $\bar{S}_0^*(z) \cdot B_1 \cdot \sigma^\circ(z)$ in the neighbourhood of the points z_0 and z_l can be non-uniform and dependent on the shape of the rod ends. Because of the localization of the δ-sequence $S(z, z', a)$ in the neighbourhood of the diagonal $z = z'$, to investigate the convergence of the integral $J(z, a)$ in the neighbourhood of the end z_0 as $a \to 0$, the initial curvilinear rod can be replaced by a rectilinear semi-infinite rod of radius $a(z_0)$ whose middle line is tangent to the curve Γ at the point z_0 and has a common origin with Γ. The asymptotic form of $J(z, a)$ in the neighbourhood of the second end $z = z_l$ can be considered similarly.

Therefore, if $a(z)$ is a function varying slowly on Γ and $a(z) \neq 0$ for $z \neq z_0, z_l$, then to analyse the asymptotic form of the right side of (3.1) as $\varepsilon \to 0$ in the neighbourhood of the end z_0 (z_l), it is sufficient to consider a rectilinear semi-infinite rod of constant radius $a(z_0)$ ($a(z_l)$) with middle line Γ_0 (Γ_l), where Γ_l is a half-line analogous to Γ_0 with origin at z_l.

In the case of a semi-infinite cylindrical rod of constant radius $|\bar{y}| < a$ in the domain V and from (3.5) there follows an expression for $\bar{S}^*(\bar{y}, k^3, a)$ in the form (we later omit the superscript 3 in the argument k^3)

$$\begin{gathered} \bar{S}^{*\alpha\beta}(\bar{y}, k, a) = -\tfrac{1}{2} c_0 [\Theta^{\alpha\beta} + 2m^\alpha m^\beta + \\ (1 - T(a|k|))(\Theta^{\alpha\beta} - 2m^\alpha m^\beta)] + c_0 P^{\alpha\beta}(|\bar{y}|k)\, T(a|k|), \\ T(a|k|) = a|k|\, K_1(a|k|) \\ P^{\alpha\beta}(t) = t^{-1}(I_1(t) - \tfrac{1}{2}t)\, \Theta^{\alpha\beta} + I_2(t)\, e^\alpha e^\beta - iI_1(t)(m^\alpha e^\beta + e^\alpha m^\beta) - \\ (I_0(t) - 1)\, m^\alpha m^\beta \end{gathered} \tag{3.6}$$

where $P(t)$ is an analytic function whose power series expansion starts with linear terms in t.

Let s be a natural coordinate along the rod axis, $s \in [0, \infty)$. Substituting expression (2.3), (2.4) for σ° on the right-hand side of (3.1) and using (3.4) and (3.6), we obtain

$$\sigma^\circ(s) - \int_0^\infty \bar{S}(\bar{y}, s-s', a)\cdot B_1\cdot\sigma^\circ(s')\,ds' = \sigma_0(s) + R(\bar{y}, s, a) \tag{3.7}$$

$$R^\alpha(\bar{y}, s, a) = {}^1/_2 c_0 B_1 (\Theta_\beta{}^\alpha - 2m^\alpha m_\beta)\,[(1-\mathbf{T})\sigma^{\circ\beta}](s) - c_0 B_1 P_t{}^\alpha(-i\,|\bar{y}|\,D)(\mathbf{T}\sigma^{\circ\beta})(s), \quad D = d/ds \tag{3.8}$$

Here $\mathbf{T}$ is the integral operator with symbol $T(a\,|\,k\,|)$ which is defined by the formula ($T(s)$ is the Fourier prototype of the function $T(a\,|\,k\,|)$ (3.6))

$$(\mathbf{T}\sigma^\circ)(s) = \int_0^\infty T(s-s')\,\sigma^\circ(s')\,ds', \quad T(s) = \frac{a^2}{2(s^2+a^2)^{3/2}} \tag{3.9}$$

Therefore, it follows from (3.1) and (3.7) that the function $\sigma^\circ(s)$ satisfies (1.2) apart from the residual R. Here and henceforth it is assumed that the external field $\sigma_0(x)$ changes slightly in the rod cross sections such that $\sigma_0(z+\rho) = \sigma_0(z)$. To estimate the quantity R we examine the asymptotic form of the function $\mathbf{T}\sigma^\circ$ as $a \to 0$.

4. The asymptotic form $\mathbf{T}\sigma^\circ$. Let $\sigma^\circ(s)$ be a smooth bounded function of the order of unity. We will estimate the result of an operator $\mathbf{T}$ defined by the relationship (3.9) acting on it by considering a to be a small quantity. We consider the following cases.

1°. The case $s \gg a$. We select A from the conditions $A \gg a$, $s - A \gg a$ and we represent the integral (3.9) in the form

$$(\mathbf{T}\sigma^\circ)(s) = \Big(\int_0^{s-A} + \int_{s-A}^{s+A} + \int_{s+A}^\infty\Big)\,T(s-s')\,\sigma^\circ(s')\,ds' \tag{4.1}$$

It can be shown that the following hold for the function $T(s)$ (3.9)

$$\int_0^{s-A} T(s-s')\,ds' = O(a^2), \quad \int_{s+A}^\infty T(s-s')\,ds' = O(a^2) \tag{4.2}$$

$$J_k(s,a) = \int_{s-A}^{s+A} T(s-s')(s'-s)^k\,ds' = O(a^2),$$

$$k \geqslant 3 \quad (k = 0, 1, 2, \ldots)$$

$$J_0(s, a) = 1 + O(a^2),\ J_1(s, a) = 0,\ J_2(s, a) = -a^2\ln a + O(a^2)$$

It hence follows that the first and third integrals in (4.1) are of the order of a^2, and to estimate the second integral we substitute therein the Taylor series expansion of the function $\sigma^\circ(s')$ in the neighbourhood of the point s

$$\sigma^\circ(s') = \sigma^\circ(s) + D\sigma^\circ(s)\,(s'-s) + {}^1/_2\,D^2\sigma^\circ(s)\,(s'-s)^2 + \cdots \tag{4.3}$$

and we use the relationships (4.2). We will consequently have

$$(\mathbf{T}\sigma^\circ)(s) = \sigma^\circ(s) - {}^1/_2\,a^2\ln a\,D^2\sigma^\circ(s) + O(a^2), \quad s \gg a \tag{4.4}$$

2°. The case $s \sim a$. We select $A \gg a$ and we represent the integral (3.9) in the form

$$(\mathbf{T}\sigma^\circ)(s) = \Big(\int_0^A + \int_A^\infty\Big)\,T(s-s')\,\sigma^\circ(s')\,ds' \tag{4.5}$$

Here the second integral is of the order of a^2 because of (4.2), and to estimate the first integral we use relationships that can be obtained by integrating by parts ($k = 0, 1, 2, \ldots$)

$$j_k(s,a) = \int_0^A T(s-s')(s'-s)^k\,ds' = O(a^2), \quad k \geqslant 3 \tag{4.6}$$

$$j_0 = 1 - \Phi_0(\zeta) + O(a^2),\ j_1 = a\Phi_1(\zeta) + O(a^2)$$

$$j_2 = -a^2\ln a\,(1 + \Phi_2(\zeta, a)) + O(a^2);\ \zeta = s/a$$

$$\Phi_0(\zeta) = \frac{1}{2}\Big(1 - \frac{\zeta}{\sqrt{1+\zeta^2}}\Big), \quad \Phi_1(\zeta) = \frac{1}{2\sqrt{1+\zeta^2}},$$

$$\Phi_2(\zeta, a) = \frac{\ln[a(1+\zeta)]}{\ln a}$$

Substituting expansion (4.3) into the first integral in (4.5) and using these equalities, we obtain the estimate

$$(\mathbf{T}\,\sigma^\circ)\ (s) = \sigma^\circ(s) - \sigma^\circ(0)\,\Phi_0\,(\zeta) + aD\sigma^\circ\,(0)\,[\Phi_1\,(\zeta) - \zeta\Phi_0\,(\zeta)] - {}^1/_2\ a^2 \ln a\,[D^2\sigma^\circ\,(s) + D^2\sigma^\circ\,(0)\,\Phi_2\ (\zeta,\ a)] + O\,(a^2),\ \ s \sim a,\ \ \zeta \sim 1 \tag{4.7}$$

It is taken into account here that $\Phi_0\,(\zeta)$ is a function of boundary-layer type localised at the edge $s=0$ as $a \to 0$, for which the following holds

$$\sigma^\circ\ (s)\ \Phi_0\,(\zeta) = \sigma^\circ\,(0)\ \Phi_0\,(\zeta) + aD\sigma^\circ\,(0)\,\zeta\,\Phi_0\,(\zeta) + O\,(a^2)$$

5. The asymptotic form $\sigma(x)$ in the neighbourhood of the rod edge. We turn to an estimate of the magnitude of the residual R in the relationship (3.7). Using (4.4) and (4.7), we will have

$$R^\alpha(\bar{y}, s, a) = {}^1/_2 c_0 B_1 (\Theta_\beta{}^\alpha - 2m^\alpha m_\beta)\,\sigma^{\circ\beta}(0)\,\Phi_0\,(s/a) + O\,(a) \tag{5.1}$$

It is seen hence and from (4.6) for $\Phi_0\,(\zeta)$ that as $a \to 0$ the function R tends non-uniformly to zero in the domain $s \geqslant 0$: in the neighbourhood of the edge $s=0$ the residual is of the order of one for any a.

To cancel the residual we add a function σ_1 dependent on the arguments $\xi = \bar{y}/a,\ \zeta = s/a$ (compare with /7/) that varies rapidly in the neighbourhood of the edge, to the function $\sigma^\circ\,(s)$ that varies slowly in scales of the order of a. Substituting the sum $\sigma^\circ\,(s) + \sigma_1\,(\xi, \zeta)$ into the left-hand side of (1.2), we transfer to the dimensionless variables $\xi,\ \zeta$ and we require that the result tend uniformly to $\sigma_0\,(s)$ in the domain $s \geqslant 0$ as $a \to 0$. We hence obtain for $\sigma_1\,(\xi, \zeta)$ the equation $(\eta = \xi + \zeta m)$

$$\sigma_1(\xi,\zeta) - \int\limits_{|\xi'|\leqslant 1} d\xi' \int\limits_0^\infty S\,(\eta - \eta')\cdot B_1\cdot \sigma_1(\xi',\zeta')\,d\zeta' = -R_0(\xi,\zeta) \tag{5.2}$$

$$R_0{}^\alpha(\xi,\zeta) = {}^1/_2 c_0 B_1 \left[\Theta_\beta{}^\alpha - 2m^\alpha m_\beta + 2P_\beta{}^\alpha\left(-i\,|\,\xi\,|\,\frac{d}{d\zeta}\right)\right]\sigma^{\circ\beta}(0)\,\Phi_0\,(\zeta)$$

where $P\,(t)$ has the form (3.6). This equation corresponds to the problem of a medium with a semi-infinite cylindrical rod of unit radius in an external field $R_0\,(\eta)$. It follows from the uniqueness of the solution of (1.2) that the field $\sigma\,(x)$ within the rod will tend uniformly to the sum $\sigma^\circ\,(s) + \sigma_1\,(\xi, \zeta)$ as $a \to 0$ since the residual R corresponding to this function uniformly on Γ for the mentioned passage to the limit.

We shall seek a solution of (5.2) that damps out at infinity in the form $\sigma_1\,(\xi,\zeta) = \sigma_1{}^\circ\,(\zeta) + \sigma_1{}^1\,(\xi,\zeta)$, by selecting the function $\sigma_1{}^\circ\,(\zeta)$ from the condition for satisfying (5.2) at points on the rod axis $(\xi = 0)$

$$\sigma_1{}^\circ(\zeta) - \int\limits_{|\xi'|\leqslant 1} d\xi' \int\limits_0^\infty S\,(\zeta m - \eta')\cdot B_1\cdot \sigma_1{}^\circ(\zeta')\,d\zeta' = -R_0\,(0,\zeta) \tag{5.3}$$

The equation for $\sigma_1{}^1\,(\eta)$ will here have the form (5.2) with right-hand side $R_1\,(\eta)$ which by virtue of (3.8) is determined by the relationship

$$R_1\,(\xi,\zeta) = -\frac{1}{2}\,c_0 B_1 P\left(-i\,|\,\xi\,|\,\frac{d}{d\zeta}\right)[(\mathbf{T}_\zeta\sigma_1{}^\circ)\,(\zeta) + \Phi_0\,(\zeta)\,\sigma^\circ\,(0)]$$

$$(\mathbf{T}_\zeta\sigma_1{}^\circ)\,(\zeta) = \frac{1}{2}\int\limits_0^\infty \frac{\sigma_1{}^\circ\,(\zeta')}{[(\zeta-\zeta')^2+1]^{3/2}}\,d\zeta' \tag{5.4}$$

The function R_1 damps out at infinity more rapidly than the right-hand side of (5.3) $(R_1 \sim \zeta^{-3}, \Phi_0 \sim \zeta^{-2}$ as $\zeta \to \infty)$, consequently, the damping of $\sigma_1\,(\xi,\zeta)$ with distance from the edge $\zeta = 0$ is determined by the function $\sigma_1{}^\circ\,(\zeta)$. We will examine (5.3) for $\sigma_1{}^\circ\,(\zeta)$ in greater detail. Integrating with respect to ξ' and using the relationships (3.2)-(3.6) we obtain two independent equations for the transverse component $\Theta\sigma_1{}^\circ$ and longitudinal component $\sigma_{1m}{}^\circ = \sigma_1{}^\circ\cdot m$ of the vector $\sigma_1{}^\circ$

$$\alpha_\Theta\Theta\sigma_1{}^\circ - \mathbf{T}_\zeta\Theta\sigma_1{}^\circ = -\Theta\sigma^\circ\,(0)\ \Phi_0,\ \alpha_\Theta = 2\ (1 + c_0B_1)\ (c_0B_1)^{-1} \tag{5.5}$$

$$\alpha_m\sigma_{1m}^\circ - \mathbf{T}_\zeta\sigma_{1m}^\circ = -\sigma_m{}^\circ(0)\,\Phi_0,\quad \alpha_m = -(c_0B_1)^{-1}$$

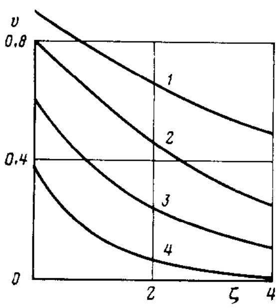
where the operator $\mathbf{T}_\zeta$ is defined by the relationship (5.4).

The parameter c_0B_1 satisfies the condition $-1 \leqslant c_0B_1 < \infty$, consequently, the domain of possible values of the coefficient α_Θ and α_m is defined by the equalities $\alpha_\Theta \leqslant 0,\ \alpha_\Theta > 2;$ $\alpha_m < 0,\ \alpha_m \geqslant 1$. The case $\alpha_m = 1$ corresponds to an absolutely "rigid" rod $(B_1 = -c_0{}^{-1}, B = 0)$.

Each of th Eqs. (5.5) is equivalent to a Wiener-Hopf equation of the form

$$(\mathbf{L}_\alpha v)(\zeta) = \alpha v(\zeta) - \frac{1}{2}\int_0^\infty \frac{v(\zeta')}{[(\zeta-\zeta')^2+1]^{3/2}}\, d\zeta' = \Phi_0(\zeta) \tag{5.6}$$

If $\alpha < 0$ and $\alpha > 1$, then the symbol $L_\alpha(k) = \alpha - kK_1(k)$ of the operator $\mathbf{L}_\alpha$ does not vanish on the real axis, and therefore, (5.6) has a unique solution in the class of continuous bounded functions /8/. The following deductions can be made from the results of a numerical solution of this equation (they are represented in the figure by curves *1-4*, corresponding to values of the parameter α equal to 1.01, 1.05, 1.2, 2.0).



1°. If $\alpha - 1 = O(1)$, then $v(\zeta)$ is analogous to $\Phi_0(\zeta)$, a function of boundary-layer type localized at the edge $s = 0$ as $a \to 0$.

2°. If $\alpha \to 1$, then the rate of decrease of the function $v(\zeta)$ as $\zeta \to \infty$ diminishes. Such a change in the nature of the solution is associated with degeneration of the symbol $L_\alpha(k)$ at the point $k = 0$ for $\alpha = 1$.

Qualitatively the solution of (5.6) behaves in the same way as the solution of

$$(\mathbf{L}_\alpha^\circ v_0)(\zeta) = \alpha v_0(\zeta) - {}^1\!/_2 \int_0^\infty e^{-|\zeta-\zeta'|} v_0(\zeta')\, d\zeta' = e^{-\zeta}$$

where $\mathbf{L}_\alpha^\circ$ is the Wiener-Hopf operator with the symbol $L_\alpha^\circ(k) = \alpha - (1+k^2)^{-1}$. It can be shown that for $\alpha < 0,\ \alpha > 1$

$$v_0(\zeta) = 2\left[\sqrt{\alpha(\alpha-1)} + \alpha\right]^{-1} \exp\left(-\sqrt{\alpha^{-1}(\alpha-1)}\,\zeta\right)$$

and the properties 1°, 2° are evident here.

Summarizing the results obtained, it can be asserted that for a rod with radius of cross-section varying slowly along the length and $1 + c_0B_1 = O(1)$ the principal term of the asymptotic form of the solution of (1.2) is represented in the form

$$\sigma^\circ(\bar{y}, s) = \sigma^\circ(s) + \sigma_{1(0)}\left(\frac{\bar{y}}{a(0)}, \frac{s}{a(0)}\right) + \sigma_{1(l)}\left(\frac{\bar{y}}{a(l)}, \frac{l-s}{a(l)}\right)$$

where the slowly varying component $\sigma^\circ(s)$ has the form (2.3), (2.4) and the functions $\sigma_{1(0)}$ and $\sigma_{1(l)}$ are determined from equations of the type (5.2) and are localized as $\varepsilon \to 0$ in a-neighbourhood of the rod ends $s \in [0, a(0)),\ s \in (l - a(l), l]$. We note that the structure of the principal term of the asymptotic form for the field $\sigma(x)$ can be different for a rod with a transverse section varying rapidly in the neighbourhood of the edges. In particular, for a rod in the shape of an elongated ellipsoid of revolution, the scheme in question results in a homogeneous equation for $\sigma_1(\xi, \zeta)$, which has only a trivial bounded solution $\sigma_1 = 0$. Therefore, in this case the principal term of the asymptotic form $\sigma(x)$ has the form (2.3), (2.4) and contains no functions of boundary-layer type.

6. A medium with a rigid rod. As $c_0B_1 \to -1$ the form of the slowly varying part of the principal term of the asymptotic form of the solution of (1.2) can differ from (2.3), (2.4) in the case of a finite rod since the edge effect zone encloses an ever greater neighbourhood of its ends as the rod "rigidity" increases (the parameter c). We will consider this case in greater detail.

We start with the construction of the formal expression for the principal term of the desired asymptotic form. Let $\sigma(s)$ be a function that is constant over the rod transverse sections. We substitute it into (1.2) and we require that this equation be satisfied at points on the middle line of the rod Γ

$$\sigma(s) - \int_0^l \bar{S}(s, s') \cdot B_1 \cdot \sigma(s')\, ds' = \sigma_0(s) \tag{6.1}$$

The kernel $\bar{S}(s, s')$ is defined by a relationship analogous to (3.2). We expand the function $\bar{S}^*$ in the representation (3.4) for $\bar{S}(s, s')$ in a formal asymptotic series in the parameter ε. Conserving terms of order $\varepsilon^2 \ln \varepsilon$ in the series for the functions I_n and K_n in (3.5), we will have $(a(s) = \varepsilon l(s))$

$$\begin{aligned}\bar{S}^{*\alpha\beta}(\bar{y}, k, a) = {}&{}^1\!/_2 c_0 [(1 + {}^1\!/_2 \varepsilon^2 \ln \varepsilon l^2(s') k^2)\Theta^{\alpha\beta}(s') - \\ &2\varepsilon^2 \ln \varepsilon l^2(s') k^2 m^\alpha(s') m^\beta(s') - ik(1 + {}^1\!/_2 \varepsilon^2 \ln \varepsilon l^2(s') k^2) \times \\ &(m^\alpha(s')\bar{y}^\beta + \bar{y}^\alpha m^\beta(s')) - 2\delta^{\alpha\beta}] + O(\varepsilon^2), \quad |\bar{y}| < a\end{aligned}$$

$$\bar{S}^*(\bar{y}, k, a) = O(\varepsilon^2), \quad |\bar{y}| > a$$

The expression for $\bar{S}(s, s')$ corresponding to this approximation of $\bar{S}^*$ takes the following form in view of (3.4):

$$\bar{S}^{\alpha\beta}(s, s') = -{}^1/_2\, c_0\, [\delta(s - s') + \varepsilon^2 \ln \varepsilon\, l^2(s')\, d^2 \delta(s - s')/ds^2] \times$$
$$\Theta^{\alpha\beta}(s') - c_0\, [\delta(s - s') - {}^1/_2\, \varepsilon^2 \ln \varepsilon\, l^2(s')\, d^2\delta(s - s')/ds^2] \times$$
$$m^\alpha(s')\, m^\beta(s') + O(\varepsilon^2)$$

where it is taken into account that $\bar{y} \to 0$ as $s \to s'$ if Γ is a smooth curve.

Substituting this expression for the kernel $\bar{S}(s, s')$ into (6.1), we arrive at the following equations for the transverse component $\Theta\sigma$ and axial component σ_m of the vector $\sigma(s)$

$$\frac{2 + c_0B_1}{c_0B_1}(\Theta\sigma)^\alpha(s) + \frac{1}{2}\varepsilon^2 \ln \varepsilon \frac{d^2}{ds^2}[l^2(s)(\Theta\sigma)^\alpha(s)] = \frac{2}{c_0B_1}(\Theta\sigma_0)^\alpha(s) \tag{6.2}$$
$$\frac{1 + c_0B_1}{c_0B_1}\sigma_m(s) - \frac{1}{2}\varepsilon^2 \ln \varepsilon\, [l^2(s)\, \sigma_m(s)] = \frac{1}{c_0B_1}\sigma_{0m}(s)$$

We seek the expression for $\sigma(s)$ in the form

$$\sigma(s) = \sigma^\circ(s) + {}^1/_2\, \varepsilon^2 \ln \varepsilon \sigma^1(s) + \ldots \tag{6.3}$$

The equation for the principal term of this expansion $\sigma^\circ(s)$ is obtained by substituting (6.3) into (6.2) and retaining components of highest order in ε on the left-hand side. Depending on the magnitude of the parameter $1 + c_0B_1 = c_0c^{-1}$ the following cases are possible.

1°. The case $c_0c^{-1} = O(1)$. Here

$$\Theta\sigma^\circ(s) = \frac{2}{2 + c_0B_1}\Theta\sigma_0(s), \quad \sigma_m{}^\circ(s) = \frac{1}{1 + c_0B_1}\sigma_{0m}(s) \tag{6.4}$$

and the expression for $\sigma^\circ(s)$ agrees with (2.3), (2.4). The connection between this function and the principal terms of the asymptotic form for the field $\sigma(x)$ within the rod is investigated below.

2°. The case $c_0c^{-1} = O(\varepsilon^2 \ln \varepsilon)$ (a rigid rod). As before, the transverse component $\Theta\sigma^\circ$ of the field σ° has the form (6.4) while we obtain the equation

$$\frac{d^2}{ds^2}[l^2(s)\, \sigma_m{}^\circ(s)] - p^2\sigma_m{}^\circ(s) = -\frac{2\sigma_{0m}(s)}{c_0B_1\varepsilon^2 \ln \varepsilon}, \quad p^2 = \frac{2(1 + c_0B_1)}{c_0B_1\varepsilon^2 \ln \varepsilon} \tag{6.5}$$

for the axial component $\sigma_m{}^\circ$ from (6.2).

The parameter p is a quantity of the order of one, consequently, the solution of (6.5) will be a slowly varying function if $l(s)$ is a smooth bounded function that does not equal zero for $s \in [0, l]$. It follows from (6.4) and (6.5) that the transverse component $\Theta\sigma^\circ$ of the vector σ° is of the order of one and it can be neglected compared with the axial component $\sigma_m{}^\circ$ which is of the order of $(\varepsilon^2 \ln \varepsilon)^{-1}$.

We will now determine the constants in the general solution of the differential Eq. (6.5). We first consider a rectilinear semi-infinite rod of constant radius. In this case, a natural condition for the determination of one of the two constants in the general solution of (6.5) is the boundedness of the function $\sigma^\circ(s)$ at infinity. We select the second constant so that the residual from the right-hand side is a minimum on substituting $\sigma^\circ(s) = \sigma_m{}^\circ(s)\, m(s)$ into the left-hand side of (1.2). It follows from (5.1) that in this case the principal term of the residual has the form

$$R^\alpha(\bar{y}, s, a) = -c_0B_1\sigma_m^\circ(0)\, m^\alpha(s)\, \Phi_0(s/a) + O(\varepsilon\sigma_m{}^\circ) \tag{6.6}$$

The component of highest order in ε in this expression obviously vanishes if $\sigma_m{}^\circ(0) = 0$. The latter condition enables us to find the second constant in the general solution of (6.5).

The components in the expression for $\sigma(x)$ that cancel the rest of the residual are of the order of $(\varepsilon \ln \varepsilon)^{-1}$ and can be discarded as compared with the principal term. The exception is the neighbourhood of the end of the rod since because of the boundary condition obtained the function $\sigma_m{}^\circ(s)$ vanishes as $s \to 0$. The form of the asymptotic form $\sigma(x)$ in the neighbourhood of the end $s = 0$ can be obtained by the same means as in Sect.5, by adding the component $\varepsilon\sigma_1(\bar{y}/a, s/a)$ to the function $\sigma_m{}^\circ(s)$. Substituting this sum into the left-hand side of (1.2) and requiring that the principal term of the residual from the right-hand side be of the order of $\varepsilon^2 \ln\varepsilon\, \sigma_m{}^\circ$, we obtain the following equation for $\sigma_1(\xi, \zeta)$

$$\sigma_1(\xi, \zeta) - \int\limits_{|\xi'| \leq 1} d\xi' \int\limits_0^\infty M(\eta - \eta')\, \sigma_1(\xi', \zeta')\, d\zeta' = R_m(\xi, \zeta) \tag{6.7}$$
$$R_m(\xi, \zeta) = c_0B_1\left[1 - P_m\left(-i\,|\,\xi\,|\frac{d}{d\xi}\right)\right][\Phi_1(\zeta) - \zeta\Phi_0(\zeta)]\, D\sigma_m{}^\circ(0)$$

$$M(\eta) = m \cdot S(\eta) \cdot m, \quad P_m(t) = m \cdot P(t) \cdot m$$

Because of the localization of the kernel $\bar{S}(s, s')$ of the form (3.2) in the neighbourhood of the diagonal $s = s'$ as $\varepsilon \to 0$, the structure of the residual R will have a form analogous to (6.6) even in the case of a curvilinear finite rod if $a(s) \neq 0, s = 0, l$. An additional component proportional to $\sigma_m^\circ(l)\Phi_0((l - s)/a(l))$, which vanishes if $\sigma_m^\circ(l) = 0$ appears here in the expression for R.

Therefore, in the case of a rigid rod ($c_0 c^{-1} = O(\varepsilon^2 \ln \varepsilon)$) the principal term of the asymptotic form of the field $\sigma(x)$ within the domain V contains only the axial component $\sigma_m(\bar{y}, s)$ which has the form

$$\sigma_m(\bar{y}, s) = \sigma_m^\circ(s) + \varepsilon\left[\sigma_{1(0)}\left(\frac{\bar{y}}{a(0)}, \frac{s}{a(0)}\right) + \sigma_{1(l)}\left(\frac{\bar{y}}{a(l)}, \frac{l-s}{a(l)}\right)\right] \tag{6.8}$$

where the function $\sigma_m^\circ(s)$ satisfies the Eq.(6.5) and the boundary conditions

$$\sigma_m^\circ(s) = 0 \qquad \text{for} \qquad s = 0, l \tag{6.9}$$

while the functions $\sigma_{1(0)}(\xi, \zeta)$ and $\sigma_{1(l)}(\xi, \zeta)$ are determined from the solution of an equation of the form (6.7) and are real only in a-neighbourhoods of the rod ends.

For a rod in the shape of an elongated ellipsoid of revolution ($a(s) = \varepsilon\sqrt{s(l-s)}$) and a homogeneous external field, a particular solution of (6.4) is the constant $\sigma_m^\circ = -2[(2 + p^2)\varepsilon^2 \ln \varepsilon]^{-1}\sigma_{0m}$. It can be shown that on substituting the vector $\sigma_m^c m$ into the left-hand side of (1.2), the residual from the right-hand side will be of the order of $\varepsilon^2\sigma_m^\circ$ for all $x \in V$. Consequently, the principal term of the asymptotic form of the field $\sigma(x)$ within the rod has the form $\sigma_m^c m$ and does not satisfy the conditions (6.9) (a result of the violation of the conditions $a(s) \neq 0, s = 0, l$).

7. An elastic medium with a curvilinear rod. In conclusion, we present the fundamental results of applying the proposed approach to the problem of a curvilinear rod in a homogeneous elastic medium. We confine ourselves to the consideration of just the slowly varying part of the principal terms of the asymptotic form of an elastic field within the rod. The scheme elucidated above for constructing these terms is carried over to the case of the theory of elasticity without change. The additional technical difficulties are associated with the higher tensor dimensionality of the functions characterising the elastic field. Later it will be convenient to use the following tensor basis to represent the quadrivalent tensors in the problem:

$$E_{1\alpha\beta\lambda\mu} = \delta_{\alpha)(\lambda}\delta_{\beta)(\mu}, \quad E_{2\alpha\beta\lambda\mu} = \delta_{\alpha\beta}\delta_{\lambda\mu}, \quad E_{3\alpha\beta\lambda\mu} = \delta_{\alpha\beta}m_\lambda m_\mu$$
$$E_{4\alpha\beta\lambda\mu} = m_\alpha m_\beta \delta_{\lambda\mu}, \quad E_{5\alpha\beta\lambda\mu} = \delta_{\alpha)(\lambda}m_{\beta)}m_{(\mu}, \quad E_{6\alpha\beta\lambda\mu} = m_\alpha m_\beta m_\lambda m_\mu$$

where the parentheses denote symmetrization over the corresponding subscripts, and m is the direction of the tangent to the rod middle line.

The integral equation for the stress tensor $\sigma(x)$ in a medium with an inclusion has the form (1.2). The tensor B_1 and the operator symbol S in (1.2) are determined in the case of isotropic media and inclusions by the relationships

$$B_1 = c^{-1} - c_0^{-1}, \quad c = \lambda E_2 + 2\mu E_1, \quad c_0 = \lambda_0 E_2 + 2\mu_0 E_1$$
$$S^*(k) = c_0 \cdot K^*(k) \cdot c_0 - c_0$$
$$K^*(k) = \frac{1}{\mu_0}\left[E_5(n) - \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}E_6(n)\right], \quad n = \frac{k}{|k|}$$

where λ_0, μ_0 are the Lamé coefficients of the **medium,** and λ, μ are the same quantities for the inclusion.

Repeating the construction scheme for the principal term of the asymptotic form of the field $\sigma(x)$ elucidated in Sect.6, we arrive at (6.1) in which the kernel $\bar{S}(s, s')$ has the following form to within terms ε^2 ($E_i = E_i(m(s))$):

$$\bar{S}(s, s') = S_0(s')\delta(s - s') - \varepsilon^2 \ln \varepsilon\, S_1(s') l^2(s') \frac{d^2}{ds^2}\delta(s - s') + O(\varepsilon^2) \tag{7.1}$$
$$S_0(s) = c_0 \cdot A_0(s) \cdot c_0, \quad S_1(s) = c_0 \cdot A_1(s) \cdot c_0$$
$$A_0(s) = \frac{1}{8\mu_0(\lambda_0 + 2\mu_0)}[2(\lambda_0 + 3\mu_0)E_1 - (\lambda_0 + \mu_0) \times (E_2 - E_3 - E_4 + 3E_6) - 4\mu_0 E_5]$$
$$A_1(s) = \frac{1}{8\mu_0(\lambda_0 + 2\mu_0)}[2\mu_0 E_1 - (\lambda_0 + \mu_0) \times (E_2 - 3E_3 - 3E_4 + 15E_6) + 6\lambda_0 E_5]$$

The equation into which (6.1) transfers on approximating the kernel $\bar{S}(s, s')$ by the right-hand side of (7.1) takes the form

$$\Pi_{0\lambda\mu}^{\alpha\beta}\sigma^{\lambda\mu}(s) - \varepsilon^2 \ln \varepsilon \Pi_{1\lambda\mu}^{\alpha\beta} \frac{d^2}{ds^2}[l^2(s)\sigma^{\lambda\mu}(s)] = \sigma_0^{\alpha\beta}(s) \tag{7.2}$$

$$\Pi_0 = c_0 \cdot (E_1 + A_0 \cdot c_1) \cdot c^{-1}, \quad \Pi_1 = c_0 \cdot A_1 \cdot c_1 \cdot c^{-1}, \quad c_1 = c - c_0$$

where the tensor components are taken in the basis of the y_1, y_2, y_3 axes with origin at the point s. It is here taken into account that the components of the tensors Π_0 and Π_1 in the basis mentioned are independent of s.

Seeking the solution of (7.2) in the form of the expansion (6.3), we obtain the following expressions for the principal term $\sigma^\circ(s)$.

1°. If $c_0 \cdot c^{-1} = O(1)$, then $\sigma^\circ(s) = \Pi_0^{-1}(s) \cdot \sigma_0(s)$.

2°. If $c_0 \cdot c^{-1} = O(\varepsilon^2 \ln \varepsilon)$, then (7.2) is equivalent to the following

$$(A_0 + c^{-1})_{\alpha\beta\lambda\mu}\sigma^{\lambda\mu}(s) - \varepsilon^2 \ln \varepsilon A_{1\alpha\beta\lambda\mu} \frac{d^2}{ds^2}[l^2(s)\sigma^{\lambda\mu}(s)] = \varepsilon_{0\alpha\beta}(s)$$

It is essential that the tensor A_0 defined in (7.1) be degenerate. This follows from the components A_{01133}, A_{02233}, A_{03333} of this tensor being zero in the basis of the axes y_1, y_2, y_3. At the same time A_1 is a non-degenerate tensor. Using the nature of the degeneration of A_0, it can be shown that the principal terms of $\sigma^\circ(s)$ of the expansion (6.3) has the form

$$\sigma^{\circ\alpha3}(s) = \sigma_m{}^\circ(s) m^\alpha(s) m^\beta(s) \tag{7.3}$$

where the scalar function $\sigma_m{}^\circ(s)$ satisfies the equation

$$\frac{d^2}{ds^2}[l^2(s)\sigma_m{}^\circ(s)] - p^2\sigma_m{}^\circ(s) = \frac{\varkappa_0}{\varepsilon^2 \ln \varepsilon}\sigma_{0m}(s) \tag{7.4}$$

$$p^2 = -\frac{E_0}{E}\frac{\varkappa_0}{\varepsilon^2 \ln \varepsilon}, \quad \varkappa_0 = \frac{1}{1+\nu_0}$$

Here E_0, E are Young's moduli of the medium and the inclusion, respectively, and ν_0 is Poisson's ratio of the medium. The boundary conditions for this equation have the form (6.9) if the function $l(s)$ does not vanish at the ends of the rod.

Let us examine an example of a rectilinear cylindrical rod of radius a and length $2l$ in a homogeneous external field of stresses σ_0. If $c_0 \cdot c^{-1} = O(1)$ then the slowly varying part of the asymptotic form of the stress field within the rod is the constant $\sigma^\circ = \Pi_0^{-1} \cdot \sigma_0$, where the tensor Π_0 is defined in (7.2). If $c_0 \cdot c^{-1} = O(\varepsilon^2 \ln \varepsilon)$, then the function $\sigma^\circ(s)$ is determined by (7.3) within which

$$\sigma_m{}^\circ(s) = \frac{\sigma_{0m}}{\delta}\left[1 - (\operatorname{ch} p)^{-1} \operatorname{ch}\left(p\left(1 - \frac{s}{l}\right)\right)\right], \quad \delta = \frac{E}{E_0}, \quad \varepsilon = \frac{a}{l}$$

and is the solution of (7.4) under the boundary conditions (6.9). An analogous expression is obtained for $\sigma_m{}^\circ(s)$ in /9/, where a variation of the method of combinable asymptotic expansions was used to solve the last equation. (The same method was used in /10/ to solve the scalar problem).

The results of Sects.5 and 6 can be used to construct the principal term of the asymptotic form of the stress field in the neighbourhood of the rod ends.

In conclusion we note that the problem of the equilibrium of an elastic medium with an inclusion in the form of a rigid rod has been considered by many authors. The first solutions of this problem for a cylindrical rod of circular cross-section and a homogeneous external field were obtained by using a number of simplifying hypotheses that are usual for "engineering" theories of elastic systems (a survey of the results in this area can be found in /11/, say). The authors of the papers mentioned proposed an equation analogous to (7.4) (for the constants l and σ_0) and boundary conditions (6.9). However, within the framework of the engineering theory, an expression is not successfully determined single-valuedly for the parameter p in (7.4) and the absence of longitudinal stresses on the rod endfaces (conditions (6.9)) is not successfully given a correct foundation since the local stress concentration can generally be **significant** in the neighbourhood of the endfaces.

Papers in which the problem under consideration was solved by perturbation methods (references to these papers can be found in /12/) a present another direction. A well-known procedure for combining the external and internal asymptotic expansions is used in these papers to construct the principal term of the asymptotic form of the solution of the problem as $\varepsilon \to 0$. Such an approach permits the single-valued determination of an expression for the parameter p in (7.4): however, as before conditions (6.9) are treated as stresses vanishing at the rod endfaces.

The method proposed in this paper enables one to examine the case of a curvilinear rod with cross-sectional radius varying along the length and an arbitrary external field without substantial complications. The problem is here reduced successfully to the investigation of the field just within the rod and this field is itself represented in a natural way in the form of the sum of slowly and rapidly varying components along the rod length. Conditions of the type (6.9) essentially acquire some other meaning: they minimize the residual from the right-hand side of the initial integral equation on substituting the general solution of (7.4) for the slowly changing components of the desired field into its left-hand side. These conditions generally depend on the shape of the ends and do not always allow of a simple physical interpretation. The initial equation can be satisfied with the necessary accuracy in the neighbourhood of the rod ends only by using the rapidly varying part of the solution for which the equations can only be solved numerically.

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